Build-Up of Dissipative Optical Soliton Molecules via Diverse Soliton Interactions

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Solitons can form bound states that are frequently referred to as soliton molecules as they exhibit molecule-like dynamics. The build-up phase of the optical soliton molecule remains elusive. Here, by means of a time-stretch technique that enables real-time access to the spectral and temporal dynamics, rich nonlinear processes involved in the build-up of soliton molecules are revealed in an ultrafast fibre laser. Specifically, the formation of closely- and well-separated bound solitons are resolved. In both cases, the build-up phases consist of three nonlinear stages including mode locking, soliton splitting, and soliton interactions. For closely-separated bound solitons, soliton interactions display wide diversities in repeated measurements, including soliton attraction, repelling, collision, vibration, and annihilation. For well-separated bound solitons, repulsive interactions dominate the soliton interactions. Numerical simulations corroborate these experimental observations. Furthermore, a conceptually different soliton molecule, the intermittent-vibration soliton molecule, is discovered and characterized. It is the intermediate state between the vibrational and stationary soliton molecules. The author's findings could assist in the understanding of the build-up phase of localized structures in different dissipative systems.

1. Introduction

Solitons, non-spreading wave structures formed by the balance between dispersion and nonlinearity, are ubiquitous in nature and are studied in many fields, including optics, Bose-Einstein condensates, and field theory.[1–8] They exhibit fascinating particle-like interactions. Solitons can attract, repel, or annihilate upon interactions.[7] Soliton interactions are one of the basic problems in soliton physics. In integrable systems, such as, those described by the unperturbed nonlinear Schrödinger equation, soliton interactions are dominated by field overlapping. As the soliton fields modify the optical properties of the medium, the propagation characteristics of a second soliton are affected.[7] In this case, the interaction potential between solitons has no local minima; therefore, bound solitons (multiple solitons bound together as a single unit) cannot be formed.[9] In dissipative systems, such as, ultrafast lasers, dissipative effects may affect soliton interactions, leading to considerably complicated dynamics.[10] Notably, dissipative effects generate oscillating structures in soliton tails that drastically alter the scenarios of soliton interactions. The oscillatory tails give rise to minima of the interaction potential and, therefore, bound solitons can be formed.[9] Recently, this binding mechanism was confirmed in different configurations of passive ring oscillators[11] and also revealed in microresonators,[12] suggesting that this binding mechanism could be universal.

Bound solitons are usually called soliton molecules[13,14] as their analogy with matter molecules is fascinating, such as, vibration, synthesis, and dissociation.[15] From the perspective of applications, soliton molecules may increase the capacity of telecommunications by providing a new coding scheme.[13] They are also attractive in all-optical information storage.[16] From the fundamental standpoint, it is crucial to explore interactions of solitons in order to understand the general dynamics of complex systems. Apart from stationary soliton molecules, there are dynamical soliton molecules in which the temporal separation or phase difference between solitons evolves with time. Recently, such time-varying soliton molecules were discovered in both Ti: sapphire and fibre lasers.[15,17]

Despite these thorough investigations on the fast dynamics of soliton molecules, a fundamental question remains elusive: how does noise evolve to a soliton molecule? In particular, the formation of a single soliton refers to the universal behavior of many nonlinear systems featuring the scenario of 'survival of the strongest'.[18] In this respect, the formation of double solitons (soliton molecules) may result from new dynamics of nonlinear systems. Soliton splitting[19] or soliton shaping of the background noise[20] was shown to be responsible for the generation of double solitons in ultrafast lasers, whereas so far only the latter scenario has been observed experimentally.[17] Other numerical studies[10,21,22] used well-separated solitons (unbound), rather than noise as the initial conditions in studying soliton molecule generation, that may overlook nontrivial nonlinear processes involved. Soliton interactions were shown to display rich dynamics during the soliton molecule formation in these simulations: they exhibit either attractive or repulsive

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interactions depending on the initial soliton separation. These abundant dynamics result from the fact that the nonlinear systems are inherently sensitive to initial conditions. Experimentally, only attractive interaction was observed during soliton molecule formation in an ultrafast Ti: sapphire laser.[17] The diversity of transient soliton interactions is yet to be demonstrated in experiments that is conducive to advance the understandings of soliton interactions in nonlinear systems. Besides, inspired by the recent observations of vibrational soliton molecules in ultrafast lasers,[15,17] it is of fundamental interest to conceptually explore different forms of soliton molecules.

In this work, we resolve the build-up of soliton molecules from noise in an ultrafast fibre laser. In particular, we describe the formation of three types of soliton molecules: ground- and excited-state soliton molecules (corresponding to the close- and wide-separation solitons, respectively), as well as a newly termed intermittent-vibration soliton molecule. We find that the formations of all these soliton molecules consist of three nonlinear stages: mode locking, soliton splitting, and soliton interactions. Our results show that the first two stages (mode locking and soliton splitting) are similar but the final stage (soliton interactions) shows different dynamics in the formations of these soliton molecules. The ground-state soliton molecule shows a variety of soliton interactions, such as, attraction, repulsion, vibration, and annihilation, whereas the excited-state and intermittent-vibration soliton molecules exhibit repulsive interactions. Note that previous numerical simulations showed that double solitons generated by soliton splitting were either unstable[19] or annihilated immediately after soliton splitting,[20] while our work shows that double solitons originating from soliton splitting can evolve to a soliton molecule. Our experiments reveal another common feature of these soliton molecules: there are two energy overshoots in the build-up phases of these soliton molecules. The intermittent-vibration soliton molecules that periodically switch between vibrational and static soliton molecules have been found for the first time.

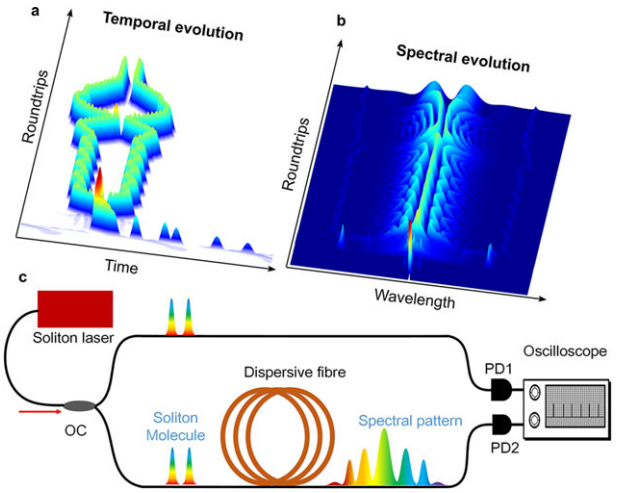

Figure 1. Probing soliton molecule formation dynamics via real-time spectroscopy. a) Sketch representation of the temporal dynamics during soliton molecule formation in a soliton laser: starting from the quasi-continuous wave, multiple pulses are generated; then one pulse survives and becomes a soliton owing to mode locking; after that, double solitons are generated from soliton splitting and, they repel and attract each other; and finally, a bound state is formed with a fixed separation and locked relative phase. b) The corresponding dynamics in the frequency domain showing the single frequency evolution to static spectral interferograms. c) The real-time spectroscopy is experimentally employed to capture the fast processes. By implementing the time-stretch dispersive Fourier transformation (TS-DFT) in the dispersive fibre, spectral interferograms are mapped into the time domain and recorded by an oscilloscope, thereby diagnosing the fast evolutions of soliton molecules. OC, optical coupler; PD, photodiode.

2. Results

2.1. Experimental Setup

As a test-bed system, we built a typical soliton fibre laser (see Figure S1, Supporting Information for the laser configuration). The laser has a cavity length of 16 meters. A sketch representation of the build-up phases of a soliton molecule is shown in **Figure 1**a,b showing the temporal and spectral evolutions, respectively. Capturing such transient dynamics in experiments is the main aim of the present work. As shown in Figure 1c, the output of the laser is split into two ports by an optical coupler (OC). One port (undispersed) is used to measure the evolution of the instantaneous intensity I(t), over a number, N, of cavity round trips (RTs) to produce a 2-dimensional spatio-temporal intensity profile I (t, N). The signal from the other port is fed into a long dispersive fibre (≈11 km in our experiments) to stretch the pulses and thus yield the spectra measurements (TS-DFT).[23] Two identical photodetectors (PD1, PD2) with a 50-GHz bandwidth were used and the signals were captured by a real-time oscilloscope with a bandwidth of 33 GHz (Agilent). It is important to note that by measuring the temporal delay between the two photodetectors (53.651 μs), we were able to conduct simultaneous measurements of the spectral and temporal intensities of the output pulses. These state-of-the-art measurement tools enabled us to resolve the transient dynamics during soliton molecule formation. The temporal and spectral resolutions of the detection system were 30 ps and 0.1 nm, respectively. During the experiment, the pump power could be turned on when the polarization controllers were preset in the settings, generating soliton molecules, and the oscilloscope was triggered to record the transient signal. We measured the formation of three different soliton molecules during the experiments, as described below.

2.2. Formation Dynamics of Different Soliton Molecules

2.2.1. Ground-State Soliton Molecule

Akin to molecules, ground-state soliton molecules have low energies, consisting of bound solitons with the shortest separation, while excited-state soliton molecules possess higher energies, corresponding to bound solitons with larger separations.[24] Experimentally, ground- and excited-soliton molecules were observed under different settings of the polarization controllers in the laser. Firstly, we obtained a ground-state soliton molecule consisting of double solitons separated by 1.8 ps that was the shortest separation obtained in the experiments and then measured its

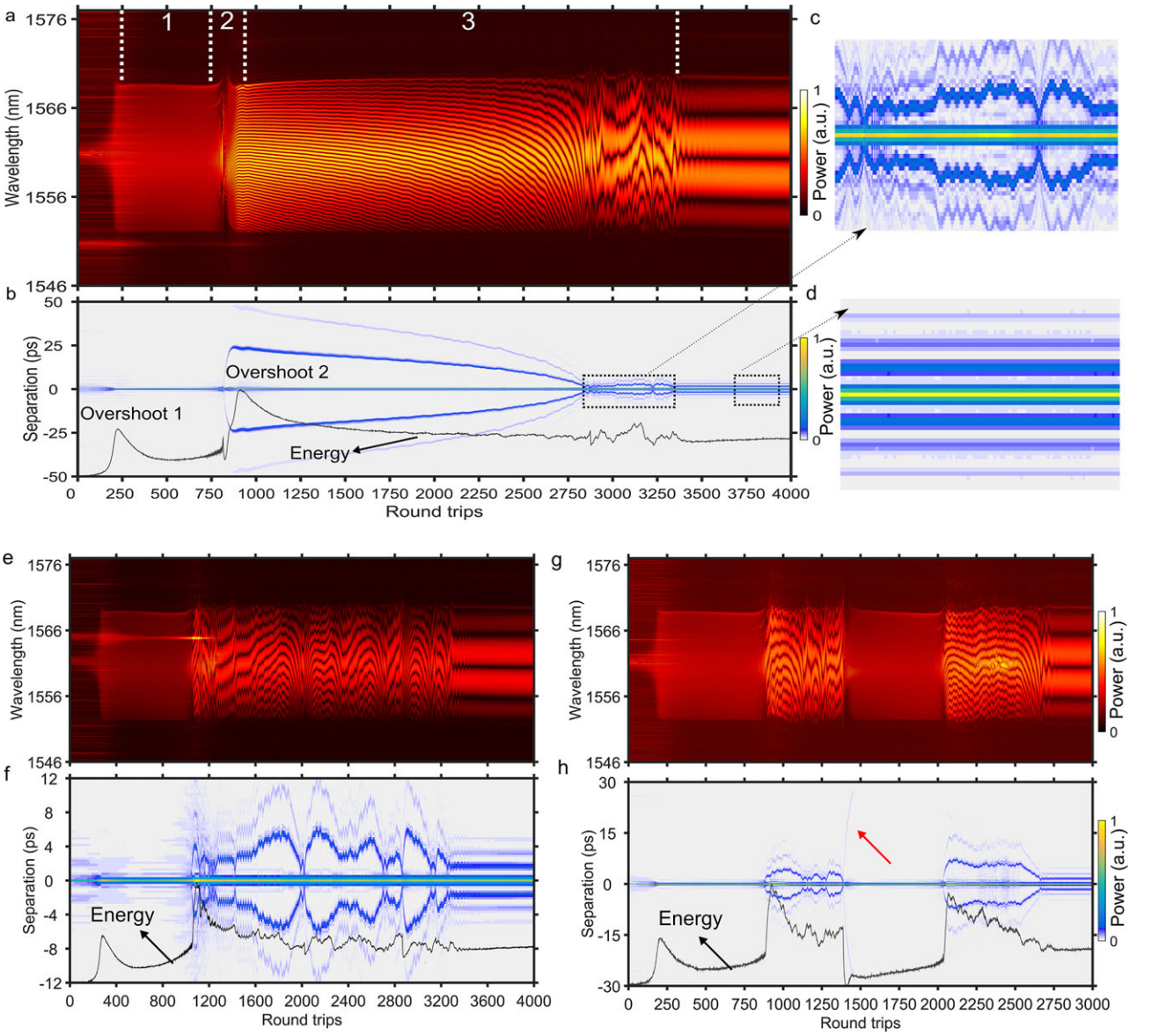

Figure 2. Ground-state soliton molecule formation from noise. a) The real-time spectra evolution measured via TS-DFT. b) The field autocorrelation traces calculated via the Fourier transforms of each single-shot spectra in (a), tracing the evolution of the temporal separations between solitons. The black line shows the energy evolution. Close-ups of the area in (b) are provided in (c) and (d), showing vibrating and stationary soliton molecules, respectively. e–h) Repeated measurements show that the formation dynamics of the ground-state soliton molecule can be considerably different: Figure 2b mainly shows attractive interactions, whereas Figure 2f,h feature vibration and annihilation, respectively.

formation dynamics. The measured TS-DFT data shown in **Figure 2**a exhibit drastic changes before a static soliton molecule is formed. Initially, up to a RT number of 250, well-defined intensity patterns occur. As the TS-DFT relies on dispersion to stretch the pulses, it only works for wide-bandwidth ultra-short pulses, and therefore these intensity patterns represent temporal information of the laser outputs. Such intensity patterns have also been observed in the build-up of mode locking.[25]

Mode locking begins near a RT number of 250, producing a wide-bandwidth signal (soliton), so the spectra can be measured by TS-DFT. The soliton lasts for ≈500 RTs (stage 1, RTs ≈250–750), before exploding into double solitons later, as indicated by the modulated spectra. These modulated spectra change for each roundtrip, indicating that soliton interactions exist within these roundtrips. These interactions can be revealed by first-order (field) autocorrelation. The Fourier transform of each single-shot spectrum yields a field autocorrelation trace according to the Wiener-Khinchin theorem. This method has been used to probe the evolving separation between bound solitons[17] and capture the transient ordering of incoherent dissipative solitons.[26] Note that if the number of pulses is n then the corresponding field autocorrelation trace has $2n$-1 peaks. The Fourier transforms of the single-shot spectra in Figure 2a provided the field autocorrelation traces shown in Figure 2b. The field autocorrelation traces

confirm that the single soliton explodes to double solitons at a RT number of ≈750 as the single peak is transformed to three peaks. The solitons repel each other as long as they are generated and the maximum separation (≈25 ps) is reached at a RT number of ≈875 (stage 2, RTs ≈750–875). Then, the solitons attract each other (decreasing in separation) until a RT number of 2750, before vibration occurs (stage 3, RTs ≈875–3250). Figure 2c shows the vibration of the solitons that is the close-up of the dashed rectangle in Figure 2b. Finally, spectral interferograms stabilize near a RT number of 3500, indicating bound solitons with fixed separations (1.8 ps) and locked relative phases are formed [see Figure 2d for close-up].

The energy evolution provides an effective way to understand the non-stationary dynamics of nonlinear systems and is used in characterizing the build-up of mode locking[25] and revealing the dynamics of soliton explosions[27–29] and rogue waves.[30] We studied the energy evolution by integrating the measured spectra over the complete spectra band, as shown in Figure 2b (black line). There are two energy overshoots in the figure. Overshoot refers to the occurrence of a signal exceeding its final value and is widely used in signal processing, electronics, and mathematics. In optics, energy overshoot occurs during the build-up of mode locking in such a way that the energy increases to a maximum value before decreasing to a steady value.[25] Here, the first energy overshoot characterizes the mode locking transition, connecting to the one in the build-up of the mode locking in Ti: sapphire lasers.[25] The second one is the result of soliton interactions. The separation between the two solitons generated from soliton splitting increases until a RT number of 875; meanwhile, the corresponding energy of the solitons increases to a maximum; then, the energy decreases while the solitons attract each other; and finally, the energy is stabilized.

As mentioned above, previous numerical simulations[10,21,22] showed that the scenarios of soliton interactions can be changed by varying the initial conditions while the generated soliton molecule is the same (a fixed point); a characteristic of many nonlinear systems. It is therefore of great interest to explore diverse soliton interactions in this transient process, given that only limited types of soliton interactions were revealed in the context of laser optics. To this end, we repeated the measurement numerous times by turning off/on the pump power, regenerating the same soliton molecules. Remarkably, the formation dynamics show diversities, as the initial condition (noise) varies randomly for each measurement. In particular, while the first two stages are similar, including mode locking and soliton splitting, the interactions between transient bound solitons (stage 3) differ significantly. Such diversities can be seen in Figure 2e–h. A periodic vibration can be seen in Figure 2f. In particular, in Figure 2h, annihilation of one of the solitons (partial annihilation) occurs near a RT number of 1500. The red arrow in Figure 2h denotes the partial annihilation processes. The separation between the two solitons increases and finally only one soliton remains. The partial annihilation of solitons were investigated in different nonlinear systems[31–33] and were also observed in optical fibre systems.[34] Such processes were observed for the first time in an ultrafast fibre laser in this study.

Only attractive interaction was discovered during soliton molecule formation in a Ti: sapphire laser.[17] In contrast, and in addition to such interactions, our work shows that a rich set of soliton interactions exist during soliton molecule formation in an ultrafast fibre laser, highlighting the difference between the two systems. In the Ti: sapphire laser, the smaller fraction of pulses shed from the first soliton are far away from that soliton by ≈1 ns, and accordingly the second soliton generated by mode locking of these fraction pulses is also ≈1 ns away from the first soliton. Hence, the subsequent interaction between the two solitons has to be attractive to form a soliton molecule, in which the final stabilized separation between the two solitons is only hundreds of femtoseconds.

2.2.2. Excited-State Soliton Molecule

Generally, if the separation between bound solitons is several soliton widths, such bound solitons are closely separated. If two solitons are spaced by a separation of an order of magnitude longer than the soliton width, such bound solitons are widely separated. Here, the excited-state soliton molecule refers to the bound wide-separation solitons. The above results (Figure 2) show that strong soliton interactions exist in the formation of a ground-state soliton molecule. The excited-state soliton molecule is expected to have weak soliton interactions, and it is necessary to know how such a soliton molecule is formed. To this end, by turning the polarization controllers, we obtained an excited-state soliton molecule with a separation of 32 ps between the two solitons, over an order of magnitude longer than that of the ground-state soliton molecule (1.8 ps). The measured TS-DFT data and corresponding field autocorrelation traces (**Figure 3**a,b) show that the first two stages are similar to those in Figure 2. The difference is that repulsive interactions (Figure 3b, near a RT number of ≈2100) dominate the final stage. Similar to the formation of the ground-state soliton molecule, the energy evolution (Figure 3b, black line) displays two overshoots. The numerical simulations suggested that repulsive interactions dominate the formation of such soliton molecules,[10,21,22] but have never been observed in experiments. Here we present the first experimental observations.

In ref. [9], it was predicted that the overlapping of oscillatory soliton tails gives rise to potential minima that accounts for the generation of bound solitons. These predictions were confirmed experimentally in passive fibre ring cavities.[11] In the context of mode-locked fibre lasers, Kelly sidebands were found to be responsible for the generation of bound solitons[35]; corresponding to oscillatory tails in the temporal domain. It is of interest to determine whether Kelly sidebands are responsible for the generation of the excited-state soliton molecule here. To this end, we checked the optical spectrum and field autocorrelation trace of the excited-state soliton molecule. Remarkably, oscillatory tails were observed in both the spectral and temporal domains, as shown in Figure 3c,d (log scale) that are the cross-sections of Figure 3a,b at a RT number of 3000, respectively. The spectral tails (Kelly sidebands) are manifestations of the dissipation terms arising mainly from the periodic perturbation by the optical coupler in the laser and are 8.7 nm away from the central spectrum (Figure 3c). The separation (8.7 nm) implies oscillatory soliton tails with a period of ≈1 ps that is compatible with those (1.3 ps) shown in Figure 3d. Note that only part of the field autocorrelation trace is shown to highlight these tails.

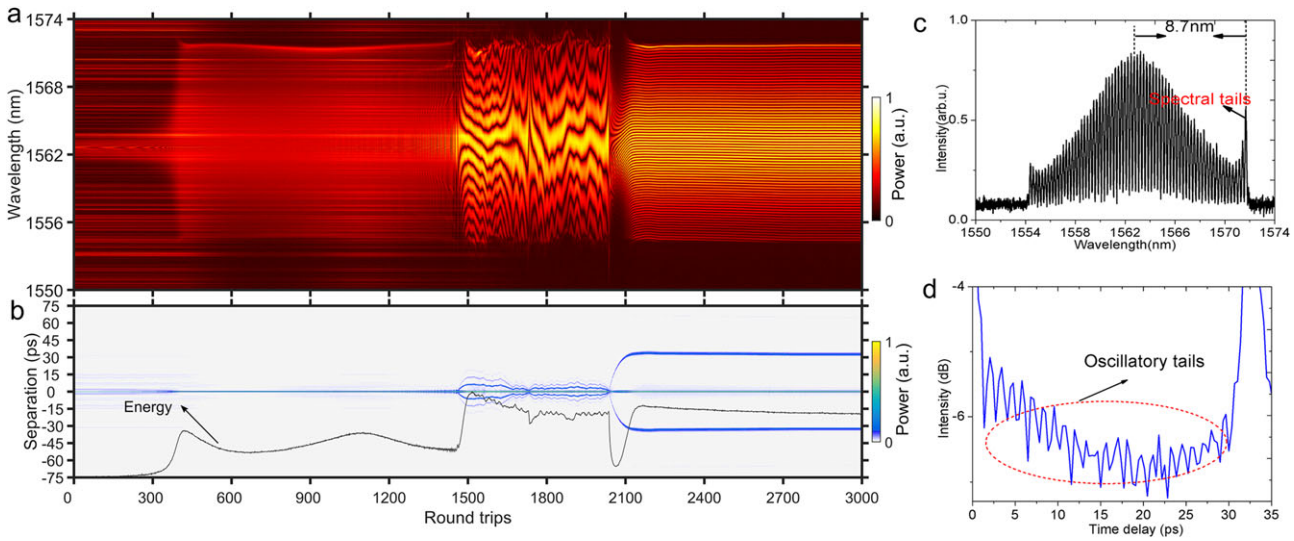

Figure 3. Formation of the excited-state soliton molecule. a) The measured TS-DFT data. b) The corresponding field autocorrelation traces. The main stages of the formation processes are similar to those of the ground-state soliton molecule in Figure 2 including mode locking and soliton splitting. However, repulsive interactions (near a RT number of ≈2100) dominate the final stage, whereas Figure 2 shows attractive interactions in the final stage. The oscillatory soliton tails are present in both the spectral (c) and temporal domains (d); (c) and (d) correspond to the cross-section of those at the last round trip (3000) in (a) and (b).

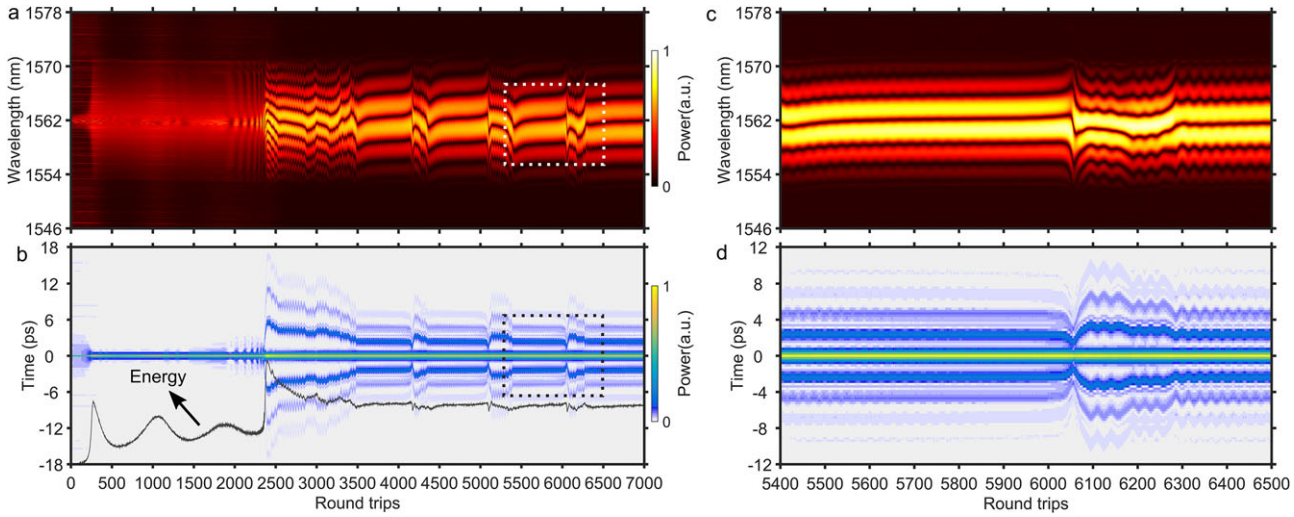

Figure 4. Dynamics of the intermittent-vibration soliton molecule. a) The measured TS-DFT data. b) The field autocorrelation traces. Close-ups of the boxes in (a) and (b), are provided in (c) and (d), respectively.

2.2.3. Intermittent-Vibration Soliton Molecule

Besides the stationary soliton molecules described above, we discover a conceptually different type of soliton molecules, termed intermittent-vibration soliton molecule. In this state, the soliton molecules periodically switch between vibrational and static soliton molecules, and is therefore the intermediate state between the two types of soliton molecules. **Figure 4**a,b show its formation dynamics and its evolution after formation. One can see that its formation stages (from 0 to ≈2500 RTs) consist of mode locking, soliton splitting, and repulsive interactions. Numerous repeated measurements show that their formation dynamics are similar. The spectra evolution of the soliton molecule is almost periodic as shown in Figure 4a. Figure 4c (close-up of the dashed box in Figure 4a) depicts the detailed evolution. The spectra are nearly static for RT numbers in the range 5400 to 6000, but vibrates afterwards, that differs from the well-known vibrational solitons the spectra of which vibrate ceaselessly.[15,17] The field autocorrelation traces shed light on the temporal evolution of the solitons. The separation between the solitons remains stable at first, before oscillating suddenly, as shown in Figure 4d (close-up of the dashed box in Figure 4b), in consistent with the spectra evolution. The separation between the solitons is 2.3 ps in the quiescent phase, close to that of the ground-state soliton molecule. By examining the energy evolution, Figure 4b (black line), there are two energy overshoots near RT numbers of 250 and 2500, respectively,

similar to the ground- and excited-state soliton molecules. In the experiment, perturbations in the fibre, polarization controllers, or optical table can force the ground-state soliton molecule to transfer to such a dynamical state.

The phase difference between the bound solitons decreases continuously from 0.95π to 0.65π in the quiescent phase from RT numbers 5400 to 6000 in Figure 4c (see Supporting Information Figure S2), before vibration occurs (oscillation in temporal separation). In contrast, the phase difference between the bound solitons in the ground-state soliton molecule is $\approx\pi$ in Figure 2 (see Supporting Information Figure S3). A numerical simulation may be useful in shedding light on the origins of the intermittent-vibration soliton molecule; this is beyond the scope of the current work.

2.3. Numerical Simulations

To provide insight into the build-up of soliton molecules, numerical simulations were investigated using a non-distributed model that considers every part of the laser. Pulse propagation within the fibre sections is modelled with a standard modified nonlinear Schrödinger equation for the slowly varying pulse envelope.

$$\frac{\partial\varphi}{\partial z} = -\frac{i}{2}\left(\beta_2 + i\frac{g}{\Omega^2}\right)\frac{\partial^2\varphi}{\partial t^2} + i\gamma|\varphi|^2\varphi + \frac{1}{2}g\varphi \tag{1}$$

Here, β_2 is the group-velocity dispersion (GVD) parameter and γ is the coefficient of cubic nonlinearity for the fibre section. The dissipative terms represent the linear gain with a Gaussian approximation for the gain profile with bandwidth Ω. The gain is described by $g = g_0 \exp\left(-\frac{Ep}{Es}\right)$, where g_0 is the small-signal gain that is non-zero only for the gain fibre, *Ep* is the pulse energy, and *Es* is the gain saturation energy determined by the pump power. To initiate and sustain the mode locking of the fibre laser, a nonlinear polarization rotation (NPR) technique was used in our experiment. In the experiments, the ground- and excited-state soliton molecules were generated under different settings of the polarization controllers indicating the transmission functions of the NPR to differ. Accordingly, we employed two different transmission functions for the two soliton molecules in the simulations (see Supporting Information Figure S4). The laser configuration used in the simulations was the same as for the experimental setup. The parameters were chosen according to the experimental values (see the parameters used in the simulation in Table S1 of the Supporting Information). The initial conditions were continuous wave with random noise. **Figure 5**a shows the simulation results referring to the formation of the ground-state soliton molecule. The scenario shows good agreement with the experimental results. Mode locking and soliton splitting are observed. In particular, the attraction of double solitons (from RT numbers of ≈500 to 3250), as well as vibration of the two solitons (from RT numbers of ≈3250 to 4250) are observed before a stationary soliton molecule is formed, supporting the experimental observations for RT numbers ≈875 to 2750 and ≈2750 to 3250 in Figure 2b. Moreover, the simulation result suggests a final separation of the ground-state soliton molecule of 2 ps that agrees well with the experimental value (1.8 ps).

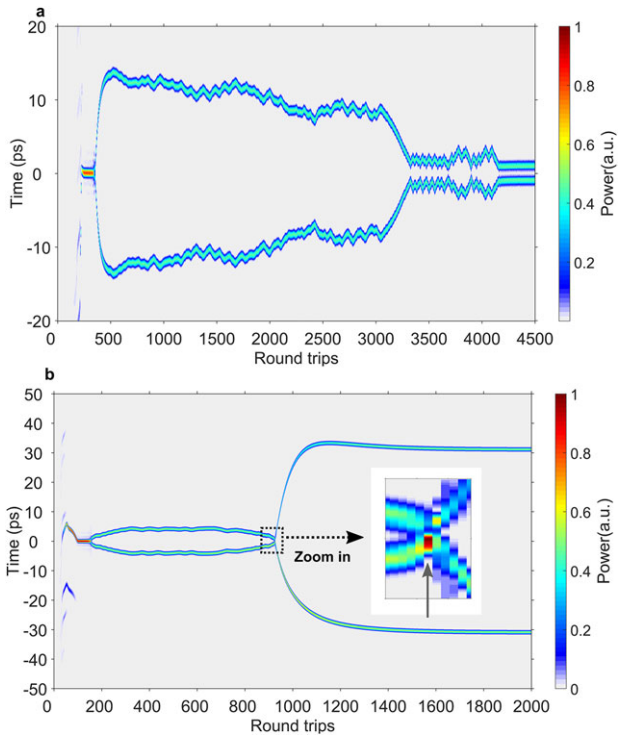

Figure 5. Simulation results of the build-up phase of the a) ground- and b) excited-state soliton molecules.

The simulation results of the formation of an excited-state soliton molecule are shown in Figure 5b. The physical processes involved are similar to the experimental observations depicted in Figure 3. The two solitons resulting from soliton splitting attract each other to a minimum separation before they repel each other to form the final soliton molecule. The magnified version of the dashed rectangle in Figure 5b illustrates that the two solitons do not merge but exhibit a significant intensity difference denoted by the grey arrow.

3. Conclusions

In this work, we have revealed the build-up phase of soliton molecules in an ultrafast fibre laser. We resolved the build-up of ground- and excited-state soliton molecules as well as a newly found intermittent-vibration soliton molecule. All of these display three nonlinear stages during their formation, including mode locking, soliton splitting, and soliton interactions. It is important to note that these transient dynamics are observed in mode-locked fibre lasers, and the build-up phase could vary in different laser systems. For example, a work showing different build-up dynamics of soliton molecules in a saturable absorber mode-locked fibre laser was posted recently.[36] The intermittent-vibration soliton molecule observed in our work is expected to be observed in a wide range of nonlinear systems, such as, ultrafast Ti: sapphire lasers, passive fibre oscillators, and microresonators.[37] Moreover, our results suggest that the gain/loss dynamics of a laser play an important role in soliton molecule build-up and thus shall also be considered in future

telecommunications when soliton molecules or multiple pulses are used as information bits, because periodic amplification and loss inherently occur in these systems. Currently, real-time measurement techniques are mainly applied to capture the dynamics of bright pulses. These techniques can also provide possibilities to investigate dark solitons and their bound states.[38]

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest

The authors declare no conflict of interets.

Keywords

mode locking, solitons, ultrafast fibre lasers

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